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Nonlinear Processes in Geophysics (2001) 8: 241–251

Nonlinear Processes in Geophysics

Diffusive draining and growth of eddies

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Received: 31 March 2000 – Accepted: 30 November 2000

Abstract. The diffusive effect on barotropic models of mesoscale eddies is addressed, using the Melnikov method from dynamical systems. Simple geometric criteria are obtained, which identify whether a given eddy grows or drains out, under a diffusive (and forcing) perturbation on a potential vorticity conserving flow. Qualitatively, the following are shown to be features conducive to eddy growth (and, thereby, stability in a specific sense): (i) large radius of curvature of the eddy boundary, (ii) potential vorticity contours more tightly packed within the eddy than outside, (iii) acute eddy pinch-angle, (iv) small potential vorticity gradient across the eddy boundary, and (v) meridional wind forcing, which increases in the northward direction. The Melnikov approach also suggests how tendrils (filaments) could be formed through the breaking of the eddy boundary, as a diffusion-driven advective process.

1 Eddies and their stability

Rings (or eddies) are significant oceanographic features which contribute considerably to fluid transport in the ocean. In particular, mesoscale (of the order of 100 km) rings formed near the Gulf Stream sometimes survive as coherent structures for periods of up to one year (Richardson, 1983). Submesoscale (of the order of 10 km) eddies may also be long-lived, and we address both mesoscale and submesoscale eddies in the present work. The observational persistence of such eddies has led to theoretical (Flierl, 1988; Helfrich and Send, 1988; Dewar and Gailliard, 1994; Dewar and Killworth, 1995; Paldor, 1999), numerical (Helfrich and Send, 1988; Dewar and Gailliard, 1994; Dewar and Killworth, 1995; Dewar et al., 1999; Paldor, 1999; McWilliams et al., 1986) and experimental (Voropayev et al., 1999) analyses of stability. Since many results indicate that eddies would tend to be *unstable*, explaining their persistence remains an active area of research. In this paper, we address a particular aspect of stability of such eddies, which reflects the effect of small diffusivity on the eddy boundary.

Though characterised by swirling fluid motions, eddies are often identified experimentally through *Eulerian* contour plots of temperature, height, salinity, or potential vorticity fields, usually obtained from two-dimensional satellite imaging data (for a review and pictures of contours, see Richardson, 1983), or from numerical schemes. Since fluid motion in the upper ocean tends to remain on surfaces of constant temperature (resp. salinity, potential vorticity, etc), rotational motion results around maxima/minima points of the appropriate scalar field, thereby forming a 'ring' (or vortical motion) in the expected sense. Often, *tendrils* (or *filaments*) are seen to emanate from these eddies, which appear to wrap around the eddy (see Fig. 4 in the experimental paper by Voropayev et al., 1999, for example).

The dynamics governing the behaviour of such eddies is assumed to be close to a two-dimensional incompressible flow in which potential vorticity is conserved (Pedlosky, 1987). Under *strict* conservation with the dynamics steady in a moving frame, no substantial deformations of eddies are to be expected, since the Lagrangian trajectories are integrable for finite times (Brown and Samelson, 1994). In many of the standard stability analyses, such a system is perturbed through an arbitrary mode, whose growth rate is determined by linearising the potential vorticity conservation equation. In this study, we adopt a different approach, which specifies the physical reason for imposing a perturbation, and also does not rely on a linearisation of the dynamics. Our perturbation shall be the result of small scale turbulence in the ocean, which is frequently modelled by a diffusive term in the governing differential equation (see Haidvogel et al., 1983, for example). The dynamics are then governed by an advection-diffusion equation for the scalar potential vorticity. Such *eddy diffusivity* has significant consequences in the advection of passive scalars, in general, fluids, and has been addressed in statistical (Poje et al., 1999), numerical (Miller et al., 1997; Poje et al., 1999) and theoretical (Fannjiang

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and Papanicolaou, 1994) senses. Bounds on the eddy diffusivity (Fannjiang and Papanicolaou, 1994; Biferale et al., 1995; Mezić et al., 1996), and descriptions of chaotic motion (Rom-Kedar and Poje, 1999; Klapper, 1992; Jones, 1994), are several features of interest. Even when *not* modelling flows with diffusivity, numerical methods often introduce a diffusivity in the interest of numerical stability, and, therefore, such numerical models could also be thought of as including eddy diffusivity effects (Rogerson et al., 1999). Unlike in regular advection-diffusion equations, the scalar quantity here is an *active* (as opposed to passive) scalar, since the potential vorticity possesses a relationship to the fluid velocity field (Pedlosky, 1987). In this study, we shall investigate how the dynamic process of eddy diffusivity affects the geometry of eddies, using a new approach, which uses elements from dynamical systems theory (Balasuriya et al., 1998), and simple geometric arguments.

Our first focus in this paper is to obtain a relationship between the growth (or decay) of such eddies, and the characteristics of the scalar potential vorticity field. Would it be possible, for example, to view the field, identify a particular eddy, and predict its chances of survival based on simple geometric properties of the scalar field? In response to this, we are able to develop a collection of (diffusivity-driven) geometric conditions for eddy growth, outlined in Sect. 5. It would be instructive to test our criteria upon available data sets with sufficient resolution. Moreover, in Sect. 7, we also obtain a qualitative condition on (small) wind forcing, which also contributes to eddy growth. 'Growth,' as specified in both these cases, will be defined through the enlargement of the eddy boundary; a shrinking boundary will correspond to a 'draining' eddy. Growing eddies have the potential of being more visible, and, therefore, are expected to be the longer lasting eddies in the ocean. Draining (shrinking) eddies, on the other hand, will eventually lose their constituent water to the ambient flow, and disappear. Therefore, in a sense, our eddy growth criteria reflect a form of eddy stability in the presence of (small) eddy diffusivity and wind forcing. It must be re-emphasised that this 'stability' is not in the traditional sense of linear stability, in which the growth rate of various modes of imposed perturbations is analysed, as in Flierl (1988); Helfrich and Send (1988); Paldor (1999); Dewar and Killworth (1995); Dewar et al. (1999).

The analysis we follow in this study is generic, and should be applicable to any system satisfying a similar advection-diffusion partial differential equation in two dimensions (for example, in tracer mixing in hydrodynamics, or in atmospheric flows). In other words, we are not using a specific model for the flows; rather, we are simply assuming that the flow satisfies the appropriate dynamical equation, and possesses the necessary kinematical properties of an eddy. These statements are made precise in Sect. 2. Section 3 then outlines the Melnikov approach from dynamical systems theory, which leads to the eddy growth criteria in Sects. 5 and 7.

A secondary goal of this paper is to give a possible explanation for the tendrils which emanate from eddies. Numerical and experimental studies, even in the laboratory rather than in the oceans, display such filaments (see Voropayev et al., 1999, for example), whose presence is certainly linked to eddy diffusivity (Robinson, 1983). Nevertheless, a geometric description of the process is lacking. Our analysis of the advection-diffusion process, from a dynamical systems viewpoint, affords an immediate and simple reasoning for the appearance of a tendril in a certain type of eddy, as explained in Sect. 6. In this case, too, it is necessary to address the deformation of the eddy boundary, which links the two aspects of this paper.

2 Dynamics

In this section, we state the mathematical equations which need to be satisfied, and also characterise the eddy boundary whose deformation is of interest. Consider a two-dimensional incompressible flow which is steady in a moving frame. This hypothesis is in keeping with many Gulf Steam models (Pierrehumbert, 1991; del Castillo-Negrete and Morrison, 1993; Pratt et al., 1995; Balasuriya et al., 1998; Weiss and Knobloch, 1989), since the Gulf Steam is steady in a gross sense in an eastward moving frame. Let (x, y) be the eastward and northward coordinates, and $\psi_0(x, y, t)$ the streamfunction of the flow. Suppose that there exists a quantity (which we shall call the potential vorticity) $q_0(x, y, t)$ which is conserved following the flow. Such is afforded in barotropic models, for example, by the quantity $\nabla^2\psi_0 + \beta y$, where β is the Coriolis parameter (Pedlosky, 1987). Depending on the particular outlook adopted, many alternative definitions of potential vorticity exist (Pedlosky, 1987), but for our purposes it suffices to think of $q_0(x, y, t)$ as *any* quantity which is preserved by the flow, and thereby

$$\frac{Dq_0}{Dt} = \frac{\partial q_0}{\partial t} - \frac{\partial \psi_0}{\partial y}\frac{\partial q_0}{\partial x} + \frac{\partial \psi_0}{\partial x}\frac{\partial q_0}{\partial y} = 0. \tag{1}$$

Particle trajectories are found by solving the ordinary differential equations

$$\dot{x} = -\frac{\partial \psi_0}{\partial y}, \qquad \dot{y} = \frac{\partial \psi_0}{\partial x}. \tag{2}$$

If q_0 is nondegenerate and smooth for all time, the flow (2) is integrable (Brown and Samelson, 1994), and complicated motion is barred. Now suppose that, in this formulation, an eddy exists, which shall be characterised as follows. At a fixed time t_0, the contours of the $q_0(x, y, t_0)$ field has a local maximum/minimum, around which closed contours exist (since q_0 is preserved by the flow, one expects the flow to remain on these contours, thereby generating the swirling motion of an eddy). This is a *continuous* model for the potential vorticity q_0, and is, therefore, somewhat different from the often used piecewise constant models (Flierl, 1988; Helfrich and Send, 1988; Paldor, 1999). Now, this entire structure would move at a constant velocity, since the flow is assumed steady in a moving frame. In other words, the contours represent streamlines, but not necessarily pathlines, and, therefore,

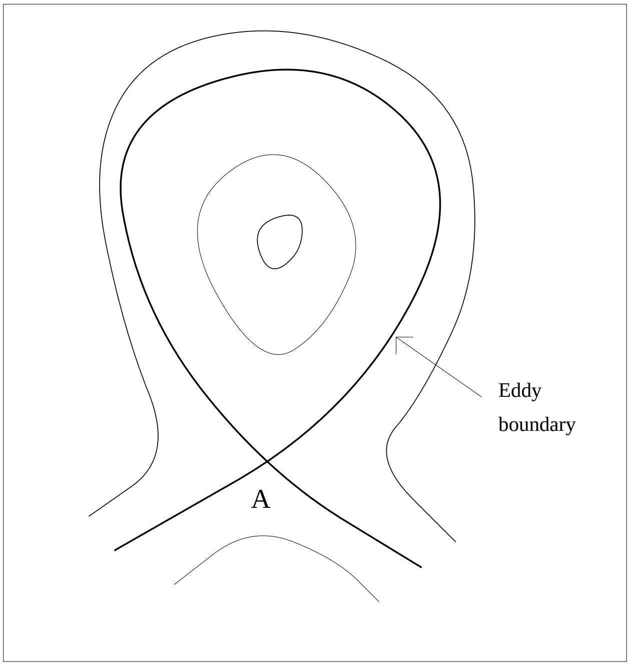

Fig. 1. Potential vorticity contours ($q_0(x, y, t_0) =$ constant).

our definition of an eddy is in an Eulerian rather than a Lagrangian sense, similar to the development in Haller and Poje (1997), who, in contrast, base their definitions on streamfunction contours, and limit the analysis to adiabatic flows. We intend to discuss the possibility of the growth of the eddy, when a suitable diffusive perturbation to the dynamics of (1) is added. To do so, we must first identify the boundary of the eddy. This shall be a contour of $q_0(x, y, t_0)$ beyond which the contour structure changes from simple closed curves to something else. It is not difficult to see that, in order for this to happen topologically for a continuous function q_0, the boundary of the eddy must contain at least one saddle point of $q_0(x, y, t_0)$. In this paper, we shall only consider exactly one saddle point A, in which case, the eddy has the structure shown in Fig. 1 (see Fig. 1 of Weiss, 1994, for a similar picture generated through a kinematic isolated eddy model). The saddle point A is a specialised point on the eddy boundary, and shall also be referred to as the *pinch-off point*. Under our present assumptions, this eddy structure would rigidly translate; no growth or shrinking can occur.

Potential vorticity conservation is, in reality, only approximately satisfied for oceanic flows (Pedlosky, 1987). We shall consider the case where the flow satisfies not the dynamics of (1), but the 'nearby' dynamics given by

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial q}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y} = \epsilon \left[\nabla^2 q + f(x, y, t) \right]. \quad (3)$$

Here, $q(x, y, t)$ is a perturbed potential vorticity, and the corresponding streamfunction is $\psi(x, y, t)$. The small positive quantity ϵ governs the size of both the diffusive term $\nabla^2 q$ and the additional forcing $f(x, y, t)$, and may be thought of as a reciprocal Péclet number. In an oceanographic context, the dynamics (3) models the presence of eddy diffusivity (averaged effect of small scale turbulence), and wind forcing, but assumes that these are both small effects in comparison to the conservation of potential vorticity (i.e. ϵ is small). Equation (3) is an advection-diffusion equation; the potential vorticity changes with time due to advection (flow of particles which have a signature potential vorticity) and also diffusion (the slow decay of potential vorticity, independent of attachment to particles). This cannot be thought of as a linear equation since the potential vorticity is linked with the streamfunction. Assuming the barotropic β-plane model (where $q = \nabla^2 \psi + \beta y$), it is possible to show that the streamfunction ψ is order ϵ close to ψ_0 (Balasuriya, 1997). Crucial to this proof is the absence of boundaries (or being far removed from boundaries); if not, this closeness may worsen to $\mathcal{O}(\sqrt{\epsilon})$, as suggested by recent results from fluid mechanics (Caflisch and Sammartino, 1998). The governing equations of particle trajectories

$$\dot{x} = -\frac{\partial \psi}{\partial y}, \quad \dot{y} = \frac{\partial \psi}{\partial x}, \quad (4)$$

has a velocity field which is, therefore, ϵ-close to that of (2). It should be noted that the 'steady in a moving frame' property has been destroyed in the perturbed flow (4); it is genuinely unsteady. The eddy boundary now perturbs: it may enlarge (a growing eddy), decrease (a shrinking eddy), or develop kinks (leading to tendrils). This geometric deformation shall be analysed using a technique from dynamical systems theory called the Melnikov method.

3 Melnikov function

To use the so-called Melnikov approach, it is first necessary to identify a fixed point and an associated homoclinic trajectory of the unperturbed fluid trajectory equation (2). Such exist if we consider the motion not in the (x, y) space, but in the moving coordinate frame in which the motion is steady. To be concrete, let us define new variables $\xi = x - c_1 t$ and $\eta = y - c_2 t$, such that the flow of (2) is steady in the (ξ, η) frame (note, that in many standard oceanographic applications, $c_2 = 0$; yet we are able to address this more general case in which eddies may propagate in an arbitrary direction, incorporating, for example, the eddies described in Dewar and Gailliard, 1994). Then, since $\dot{\xi} = \dot{x} - c_1$ and $\dot{\eta} = \dot{y} - c_2$, we have

$$\begin{aligned} \dot{\xi} &= -\frac{\partial}{\partial \eta}\left[\Psi_0(\xi, \eta) + c_1 \eta - c_2 \xi\right] \\ \dot{\eta} &= \frac{\partial}{\partial \xi}\left[\Psi_0(\xi, \eta) + c_1 \eta - c_2 \xi\right]. \end{aligned} \quad (5)$$

Notice that, since the flow must be steady in this frame, Ψ_0 has no explicit t-dependence, and, therefore, can be represented purely in terms of (ξ, η). We are adopting the convention that a capital letter denotes the variable with respect to the (ξ, η) coordinates. We see that $\Psi_0(\xi, \eta) + c_1 \eta - c_2 \xi$ serves as an effective streamfunction in the moving frame. It is not difficult to show using (1) expressed in the (ξ, η) coordinates that $Q_0(\xi, \eta)$ is functionally related to this effective streamfunction; the flow is confined to the curves $Q_0(\xi, \eta) =$ constant (or equivalently, curves where the effective streamfunction is constant). An incidental observation, which shall become important later, is that the spatial derivatives ∇, ∇^2, etc, remain invariant under the transformation from (x, y) to (ξ, η).

Now in the (ξ, η) frame, the eddy illustrated in Fig. 1 exists as a *steady* object, and, therefore, the special point A is, in fact, a fixed point of the flow (5). Additionally, it is a saddle point of the Q_0 scalar field, and hence, ∇Q_0 is zero at A. With no loss of generality, we shall choose the origin $(0, 0)$ of the (ξ, η) system to be precisely at the point A. Note the presence of a specialised trajectory of (5), which approaches the origin in forward and backwards time. This is a *homoclinic* trajectory; (a branch of) the unstable manifold of the fixed point coinciding with (a branch of) its stable manifold. The stable manifold (denoted W^s) is the set of points which asymptotically approaches the fixed point in forward time, while the unstable manifold (denoted W^u) does so in backward time. The homoclinic precisely defines the eddy boundary, and, therefore, the growth of the eddy is affected by how this homoclinic trajectory perturbs. Now the homoclinic trajectory can be represented by $\left(\bar{\xi}(t), \bar{\eta}(t)\right)$, parametrised by time t, as shown in Fig. 2. At each point $P(t) = \left(\bar{\xi}(t), \bar{\eta}(t)\right)$, one can draw a normal $N(t)$ to the eddy boundary, whose direction is given by the vector $\nabla Q_0\left(\bar{\xi}(t), \bar{\eta}(t)\right)$, which points either into or out of the eddy (this direction remains consistent on the homoclinic). This vector decays to zero as $t \to \pm\infty$; i.e. as the origin is approached. We now address how the homoclinic trajectory, which forms the eddy boundary, perturbs under the dissipative perturbation given by (3).

Suppose the solution to (3) is given in terms of the moving frame coordinates by $\Psi(\xi, \eta, t)$. An explicit t-dependence exists in this perturbed streamfunction, since the flow is no longer steady in the moving frame. However, Ψ and Ψ_0 differ only by $\mathcal{O}(\epsilon)$. The relevant particle trajectories in the moving frame are obtained through solving

$$\begin{aligned} \dot{\xi} &= -\frac{\partial}{\partial \eta}\left[\Psi(\xi, \eta, t) + c_1\eta - c_2\xi\right] \\ \dot{\eta} &= \frac{\partial}{\partial \xi}\left[\Psi(\xi, \eta, t) + c_1\eta - c_2\xi\right]. \end{aligned} \tag{6}$$

In contrast with the unperturbed moving frame equation (5), equation (6) has explicit t-dependence. Its phase space, then, is three-dimensional, and given by the variables (ξ, η, t). The fixed point at the origin perturbs to a trajectory in this three-dimensional phase space, which remains $\mathcal{O}(\epsilon)$ close to the line $(\xi, \eta) = (0, 0)$. The associated stable and unstable manifolds of this trajectory, two-dimensional in this three-dimensional phase-space, also persist. The proofs of these two claims follow from theoretical results from dynamical systems (Hirsch et al., 1977; Fenichel, 1971). The key point, however, is that there is no reason for the stable and unstable manifolds to coincide any more. Imagine that we have

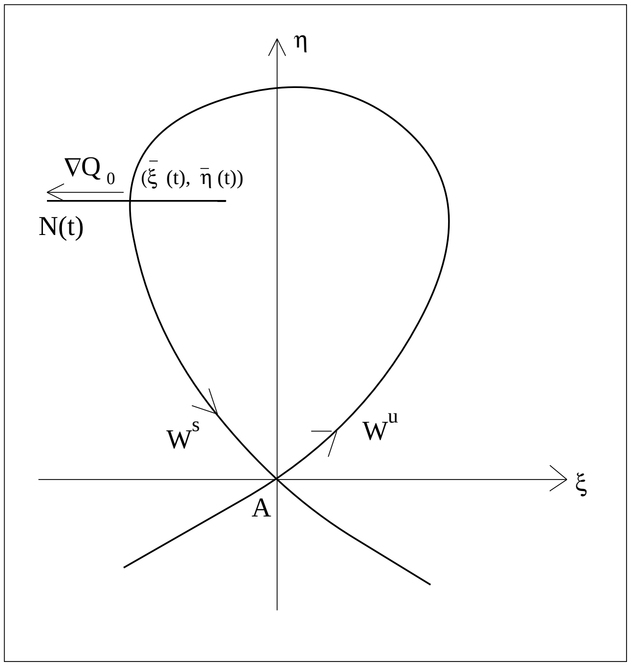

Fig. 2. The unperturbed eddy in the (ξ, η) moving frame.

intersected the three-dimensional phase-space with a plane $\{t = \text{constant}\}$. If unperturbed (if $\epsilon = 0$), one obtains exactly the picture of Fig. 2 in each and every t-slice. When $\epsilon \neq 0$, on the other hand, a generic picture of the form of Fig. 3 is formed. The perturbed manifolds W^s and W^u labelled in Fig. 3 are, in reality, the intersections of the two-dimensional manifolds with the $t =$ constant time-slice. The Melnikov approach provides a method of measuring the distance between the perturbed manifolds in this time slice of the phase-space. Consider any point $P(t) = \left(\bar{\xi}(t), \bar{\eta}(t)\right)$ on the unperturbed homoclinic (sketched as a dashed curve in Fig. 3), and think of measuring the distance $d(t)$ between the perturbed manifolds *along the normal* $N(t)$ drawn at $P(t)$. This $d(t)$ shall be a *signed* distance, whose sign is allocated as follows. If the vector drawn from the perturbed stable manifold W^s to the perturbed unstable manifold W^u is in the direction of $\nabla Q_0\,(P(t))$, then a positive value is assigned; if in the opposite direction, a negative value is given. Notice that if $d(t) = 0$, there is an intersection between these manifolds at $P(t)$, which may result in complicated mixing across the eddy boundary near $P(t)$ due to *homoclinic tangling*. It turns out that $d(t)$ can be expressed as

$$d(t) = \epsilon \frac{M(t)}{\left|\nabla Q^0\,(P(t))\right|} + \mathcal{O}(\epsilon^2), \tag{7}$$

where $M(t)$ is the *Melnikov function* (for more details, see the standard reference Guckenheimer and Holmes (1983)). In an intuitive sense, one may think of (7) as being a Taylor expansion of the distance with respect to the small parameter ϵ, whose leading order term involves the Melnikov function. Thus, for small ϵ, $M(t)$'s behaviour essentially governs the splitting between W^s and W^u at $P(t)$ (the denominator of

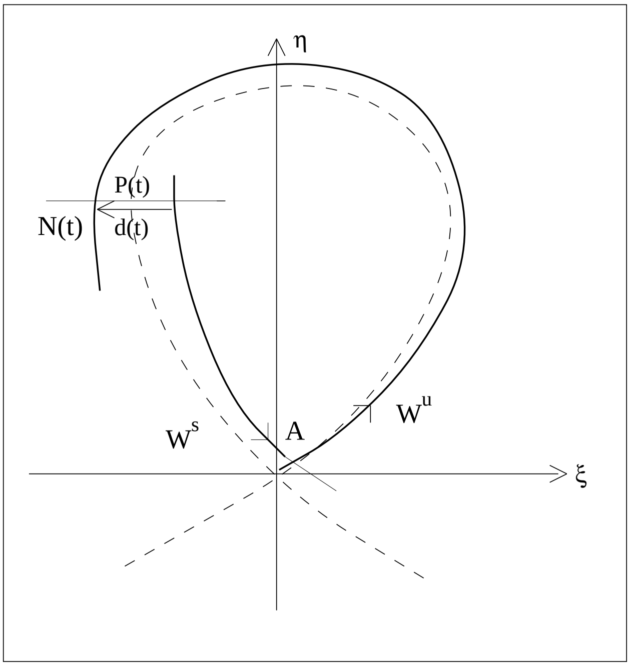

Fig. 3. The perturbed eddy in the (ξ, η) moving frame.

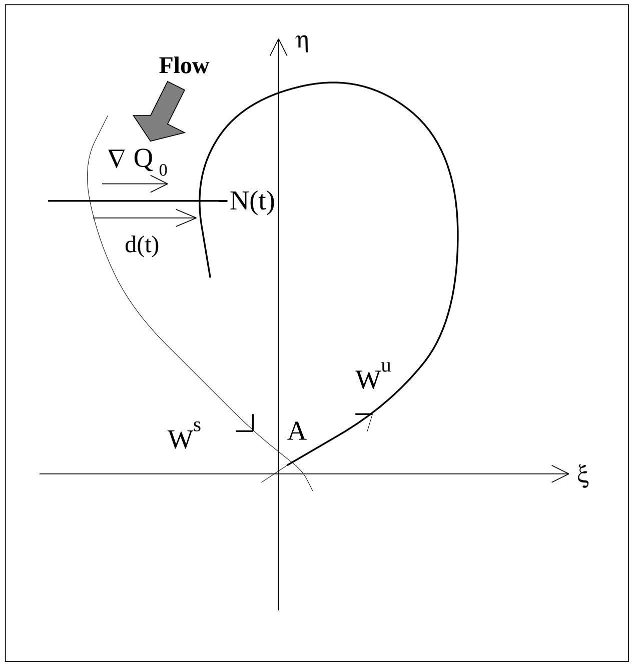

Fig. 4. A warm eddy with $M(t) > 0$.

the $\mathcal{O}(\epsilon)$ term is nonzero for all t, though it approaches zero as $t \to \pm\infty$).

We now write an expression for $M(t)$ which results directly from using the dynamical equations (5) and (6), and which was developed in a slightly different context by Balasuriya et al. (1998). Their analysis pertains to cats-eyes regions adjacent to oceanic jets, and the possibility of fluid from the jet core escaping to retrograde regions. Nevertheless, a similar approach works for the Eulerian eddies of this paper, with only slight modifications necessary to the original proof in Balasuriya et al. (1998). In the present setting, we simply state that the Melnikov function $M(t)$ can be expressed as

$$M(t) = M_d + M_f(t), \tag{8}$$

where the diffusive contribution M_d is given by

$$M_d = \int_{-\infty}^{\infty} \left[\nabla^2 Q_0 \left(\bar{\xi}(\tau), \bar{\eta}(\tau) \right) - \nabla^2 Q_0(0,0) \right] \mathrm{d}\tau, \tag{9}$$

and the forcing contribution $M_f(t)$ by

$$M_f(t) = \int_{-\infty}^{\infty} \left[F \left(\bar{\xi}(\tau), \bar{\eta}(\tau), \tau + t \right) - F\left(0, 0, \tau + t\right) \right] \mathrm{d}\tau. \tag{10}$$

Here, $F(\xi, \eta, t) = f(x, y, t)$, following our standard notation of using a capital variable to denote quantities in the moving coordinates. For details of the derivation of these results, the interested reader should follow the original proof (Balasuriya et al., 1998), making appropriate corrections for the differing geometry provided by eddies. The power of these expressions is that no knowledge of the perturbed velocity field is required; the Melnikov function can be represented only in terms of quantities related to the unperturbed flow and the wind forcing. We will now use these results to derive conditions which specify whether a given eddy grows or shrinks under the diffusive perturbation.

4 Warm and cold eddies

It is necessary to first identify two types of eddies which are seen in the ocean: warm-core and cold-core eddies. With respect to the Gulf Stream, eddies which split off from the Gulf Stream and meander onto the colder northern side are *warm-core eddies*, since they contain waters from the warmer southern oceans (Richardson, 1983). On the other hand, cold-core eddies split off towards the southern side of the Gulf Stream. Temperature, being a measure of energy, is related to potential vorticity, and hence, we shall *define* a warm eddy to be an eddy in which the potential vorticity in the interior takes on a higher value than that of the exterior; a cold eddy is defined analogously.

The principal difference between a cold and warm eddy is the direction of the vector ∇Q_0 on its boundary. For a warm eddy, Q_0 increases towards the eddy's interior, and hence, the vector $\nabla Q_0\,(P(t))$ will point inwards at every point on the eddy boundary. Now, consider the case where $M(t)$ is positive at some value of t. Then, by (7), $d(t) > 0$ for sufficiently small ϵ. This implies that the vector from W^s to W^u along $N(t)$ lies in exactly the same direction as $\nabla Q_0\,(P(t))$; i.e. it points inwards. Thus, W^u will lie *inside* W^s near $P(t)$. This is illustrated in Fig. 4, with the splitting (in reality of

$\mathcal{O}(\epsilon)$) exaggerated for clarity. Looking at the direction of flow along W^s and W^u, it is clear that the flow in the channel between them shall also be inwards. This causes fluid from the outside to flow into the eddy. The flow remains incompressible, and, therefore, the eddy compensates by growing in size with time. Thus, if $M(t) > 0$ for a warm eddy, it will grow under the diffusive perturbation. On the other hand, if $M(t) < 0$, W^s will lie inside W^u, and water would drain out of the eddy, causing it to shrink. For a cold eddy, these conditions are exactly reversed. Growing eddies are inclined to be stable under a diffusive perturbation, while shrinking eddies can be thought of as unstable, since they will drain out all their constituent water and thereby disappear.

Now, suppose we can determine conditions under which $M(t) > 0$ for a warm eddy. Since this will lead to a growing eddy, it makes sense to think of these as *stability criteria*, causing the eddy to be more visible for a long period of time. It must be emphasised that this is not stability in the conventional sense, but in the sense of leading to eddy prominence in the presence of small diffusion and forcing. We note that, by our arguments, *positive* terms in the Melnikov function contribute towards eddy growth (stability) for a *warm* eddy. Alternatively, *negative* terms are associated with *cold* eddy growth (stability). We shall now look for terms in the diffusive and forcing contributions of the Melnikov function, which provide the appropriate sign for eddy growth. Our goal is to determine criteria which are universally valid for both warm and cold eddies.

5 Diffusive criteria

With no loss of generality, we shall, in this section, assume that $Q_0(0,0) = 0$. If not, we can simply add the necessary constant to Q_0 to make it so; the dynamical equations remain satisfied since they only depend on derivatives of Q_0. The unperturbed eddy boundary is then a portion of the level curve $Q_0 = 0$. From (9), we see that the diffusive contribution to the Melnikov function, M_d, is a constant. This value is, therefore, independent of the location $P(t)$ on the homoclinic at which the splitting distance is to be measured. We note that the integrand of (9) contains two terms which relate to the Laplacian of Q_0 at an arbitrary point $P(\tau)$ on the homoclinic, and at the origin.

Let us first focus on the value of $\nabla^2 Q_0(0,0)$. Since the origin is a saddle point of Q_0, a local expansion of Q_0 near the origin does not contain terms linear in (ξ,η). The leading order terms are quadratic, and moreover, we can choose our axes such that no mixed quadratic term appears. It may be necessary to rotate the (ξ,η) coordinate system rigidly to do so, which can be done with impunity, since the Laplacian is invariant under a rotation of coordinates. Effectively, we are choosing coordinates in such a fashion that the η-axis points directly into the eddy (locally at the origin), such that it bisects the angle formed by the tangents to W^s and W^u (the *global* picture of the eddy need *not* have symmetry with respect to the η-axis). Now, the coefficients of ξ^2 and η^2 must have opposite signs, since the origin is a saddle. Thus, to leading order, we can express Q_0 near the origin by

$$Q_0(\xi,\eta) = k\left(\xi^2 - a^2\eta^2\right),$$

where k and a are some constants. The local Q_0 contours are clearly hyperbolic. The lines $\xi = \pm a\eta$ constitute the level 'curve' $Q_0 = 0$. Thus, the actual eddy boundary (itself, part of the set $Q_0 = 0$) locally can be represented by portions of these two lines. In other words, the eddy boundary is tangential to $\xi = \pm a\eta$ at the origin, (see Fig. 5). We shall define the *pinch angle*, θ, of the eddy to be the angle subtended at the pinch-off point (the origin) by the eddy boundary. Elementary calculus gives $\tan(\theta/2) = a$, and, therefore, near the origin,

$$Q_0(\xi,\eta) = k\left(\xi^2 - \tan^2(\theta/2)\,\eta^2\right).$$

By taking the Laplacian derivative,

$$\begin{aligned}\nabla^2 Q_0(\xi,\eta) &= 2k\left(1 - \tan^2(\theta/2)\right)\\ &= 2k\cos\theta\sec^2(\theta/2)\,. \end{aligned} \tag{11}$$

The Laplacian at the origin can, thus, be quantified in terms of the pinch angle and the strength of Q_0 (measured by k). Notice also that for this choice of Q_0,

$$\frac{\partial Q_0}{\partial\eta} = -2k\tan^2(\theta/2)\,\eta.$$

For a warm eddy, Q_0 must increase as one proceeds from the origin in the positive η direction, and, therefore, $k < 0$ corresponds to a warm eddy (similarly, $k > 0$ is a cold eddy). For the moment, imagine that the eddy is warm. Now, the pinch angle θ satisfies $0 \le \theta \le 180^0$, and by inspection of the trigonometric function in (11), we can see that $\nabla^2 Q_0(0,0)$ is negative if $\theta < 90^0$. Since a negative sign appears in front of $\nabla^2 Q_0$ in the expression (9) for M_d, this means that if $\theta < 90^0$, a positive contribution to the Melnikov function results. Since, for a warm eddy, a positive Melnikov function was argued to be stabilising in Sect. 4, *pinch angles less than* 90^0 *contribute towards eddy growth*. This statement is actually independent of whether a warm or cold eddy is chosen, as a similar analysis of a cold eddy would confirm. Acute pinch angled eddies are better equipped to survive than obtuse pinch angled ones.

The Laplacian at an arbitrary point P also appears in (9). In order to determine the sign of this quantity, we adopt a moving coordinate system. At each point P on the eddy boundary, let O be the centre of curvature of the homoclinic. Note that locally, the homoclinic is a circular arc near P, whose centre is at O, and radius is the radius of curvature R, (see Fig. 6). We shall make the simplifying assumption that the eddy is *convex*; O will always lie towards the interior of the eddy from the perspective of P. We choose polar coordinates (r,ϕ) attached to O, where ϕ is measured with respect to the line OP. Thus, P has coordinates $(r,\phi) = (R,0)$, and we will think of Q_0 as also expressed in these coordinates. The Laplacian operator is independent of the choice

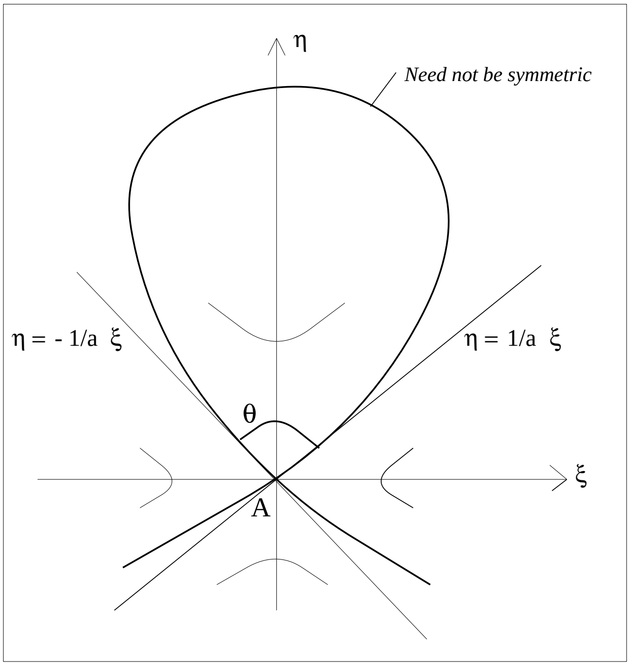

Fig. 5. Calculating $\nabla^2 Q_0$ near the origin.

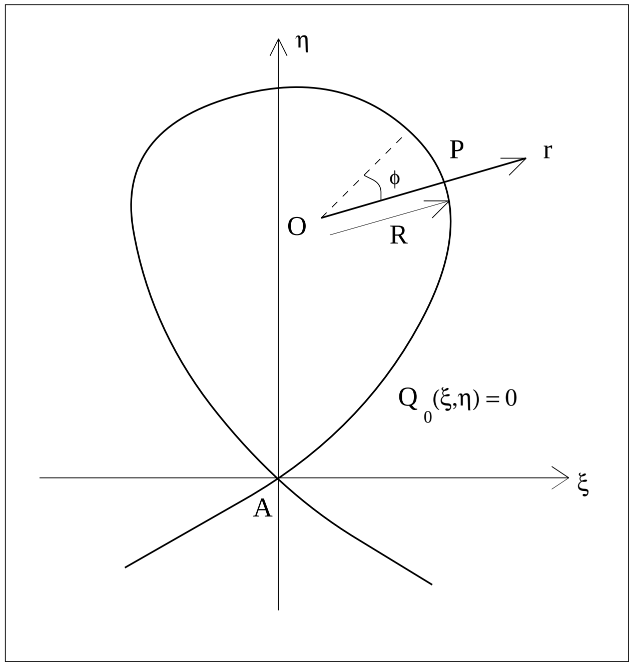

Fig. 6. Calculating $\nabla^2 Q_0$ at a point on the homoclinic.

of coordinates, and hence, we can use its polar coordinate representation

$$\nabla^2 Q_0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial Q_0}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 Q_0}{\partial \phi^2}.$$

When evaluating at $P \equiv (R, 0)$, since the homoclinic is locally a circular arc near P which is representable by $Q_0 = 0$, we see that the ϕ derivative does not contribute (Q_0 does not change when ϕ is varied). Thus,

$$\nabla^2 Q_0(P) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial Q_0}{\partial r}\right)\Bigg|_{(R,0)}$$
$$= \frac{\partial^2 Q_0}{\partial r^2}(R, 0) + \frac{1}{R}\frac{\partial Q_0}{\partial r}(R, 0)$$

The r partial derivative is the derivative in the normally outward direction from the eddy. With an abuse of notation, we shall represent the above quantity as $Q_0'' + Q_0'/R$, with the understanding that the dash is the partial derivative with respect to r, and that everything is evaluated at $(r, \phi) = (R, 0)$. We caution that P, O, R, r and ϕ are themselves dependent on τ, the parametrisation along the homoclinic.

Once again, consider a warm eddy. Recalling that positive terms of the Melnikov function contribute towards eddy growth, and noticing from (9) that $\nabla^2 Q_0(P)$ appears as a positive term, we would like to list geometric conditions which provide positive contributions from $Q_0'' + Q_0'/R$. For a warm eddy, $Q_0' < 0$, and hence, the second term, contributes the wrong sign. *Large potential vorticity gradients in the cross-eddy direction are detrimental to eddy stability*. Observe, however, that this effect is mitigated if R is large. *Eddies which have larger radii of curvature are less inclined to shrink*. This statement is also true if one considers a cold eddy.

In some senses, larger eddies will automatically have larger radii (if one is comparing eddies which have the same shape, but differ only in scale). Thus, *larger eddies leak less than smaller ones*, all other factors being equal. This statement appears to be at odds with linear stability analysis, which suggests that larger eddies are more unstable (Flierl, 1988; Helfrich and Send, 1988). There are two reasons why our results do *not* contradict these papers. Firstly, we are addressing a specific form of perturbation that involves a diffusive term in the dynamical equations (in contrast with a linear stability analysis on a non-diffusive equation). Secondly, our model is barotropic, while baroclinic instabilities are the dominating features in the cited studies. In fact, in a sequence of papers, Dewar makes a strong case for Gulf Stream eddies to be 'barotropically dominated', meaning that the flow in lower layers of the ocean so strongly follows the surface flow, thus making baroclinic instability the 'wrong' mechanism to examine (Dewar and Gailliard, 1994; Dewar and Killworth, 1995; Dewar et al., 1999). Comments that strongly pro-rotating lower flows *may* improve stability also appear in Flierl (1988); Helfrich and Send (1988). Finally, we note that *observationally*, the Gulf Stream has many large eddies which are long-lived, lending credence to our claim that larger eddies are more stable (to diffusivity).

It remains to address the term Q_0''. Suppose the function $Q_0(r, 0)$ is plotted versus r. The graph cuts the r-axis at $r = R$, and its slope is negative here for a warm eddy. The concavity of this graph at this point represents the sign of Q_0''. If concave up, we have $Q_0'' > 0$: the correct contribution towards eddy growth. This means that potential vorticity gradients are larger just inside the eddy than outside. *If potential vorticity contours are more tightly packed just inside the eddy (in comparison to just outside), this contributes towards eddy growth*. In effect, stronger potential vorticity gradients just inside the eddy (in comparison to just outside), provide a protection to the eddy waters. By addressing a cold eddy and arguing analogously, it can be shown that this qualitative requirement also holds for a cold eddy.

To summarise, we have shown that the following geometric features are conducive to (warm *or* cold) eddy growth under small diffusivity:

– Pinch angle $\theta < 90^0$

– Large radius of curvature R

– Small potential vorticity gradient Q_0' across eddy boundary

– Potential vorticity contours more tightly packed inside than outside the eddy

The opposite of each of these statements can be considered contributory factors towards depletion of eddies. It must, however, be noted that, in reality, the Melnikov function of (8) and (9) includes suitably *integrated* functions, whereas we have only addressed the sign of the integrand. It is quite conceivable that the integrand take positive and negative values, while the integral is positive (say) for a warm eddy, and, therefore, it will grow. In other words, the conditions we have stated must be taken as *qualitative*, based simply upon the contributions towards the Melnikov function taking the appropriate sign, rather than wholly describing the Melnikov function. Additionally, it must be cautioned that the eventual behaviour of the eddy is governed by the *combined* effect of the conditions; a judgment may be impossible based on only, for example, the pinch angle. Furthermore, as the eddy grows or shrinks, its geometry will change dynamically, resulting in changed behaviour.

An important consideration in using these qualitative conditions comes from the fact that, when using ocean data, what we have is the *perturbed* picture corresponding to the dynamics (3), rather than the unperturbed (1). In other words, the picture we see will not be that of Fig. 2, but a perturbation of this. The potential vorticity field that we observe would be perturbed rather than unperturbed. Hence, in applying the conditions, we are forced to rely on Q contours as opposed to Q_0 contours. Since Q is a small perturbation of Q_0, we would expect some closeness in the contours, validating this approach.

6 Tendril formation

In addition to giving criteria on eddy growth, the Melnikov function enables us to qualitatively explain the presence of a tendril emanating from an eddy with one saddle point on its boundary. Tendrils often appear in the potential vorticity (or relevant scalar field) contours, as thin lobes which

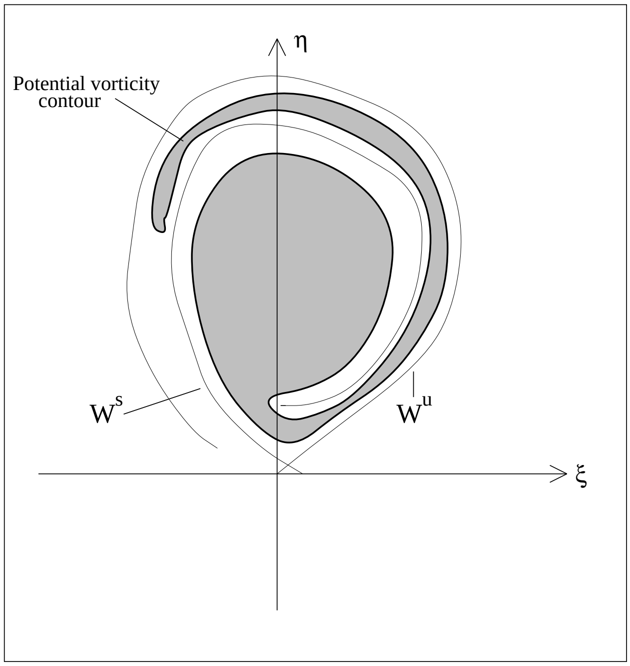

Fig. 7. A tendril of an eddy.

wrap around an essentially convex eddy structure (the experimental paper by Voropayev et al. (1999) shows some tendril structures). The presence of a tendril can be explained as a direct consequence of the fact that the diffusive contribution to the Melnikov function, M_d, is constant. If the forcing contribution is momentarily ignored, the Melnikov function would itself be constant, meaning that it is independent of the choice of the point $P(t)$ at which the measurement between W^u and W^s is made. Now, since this constant is nonzero generically, this implies that W^s and W^u do not intersect for any choice of $P(t)$; i.e. near any point on the homoclinic. Whether M is positive or negative, the consequence of constancy is a thin channel which opens up along the boundary of the eddy, as shown in Fig. 7. Fluid flows along this, in the basic direction of the flow on the manifolds, which causes the eddy to either grow or drain. In either case, however, transport occurs between the interior and exterior waters, which gradually homogenises the potential vorticity. Thus, interior waters would have potential vorticity values close to the values along this channel. If viewing potential vorticity contours, this should be visible as a tendril, exactly as observed physically. Fig. 7 shows how an Eulerian potential vorticity contour might appear in the presence of fluid transport of this nature.

The distance expansion (7) shows that a tendril's width would be of $\mathcal{O}(\epsilon)$. The eddy entrains (resp. drains out) water along the tendril, which forms the 'feeding' (resp. 'excreting') organ of the eddy. The fluid velocities in the tendril are not small (they are $\mathcal{O}(1)$ rather than $\mathcal{O}(\epsilon)$), and hence, the tendril stretches at a more rapid rate than the diffusive decay of the eddy on the whole. Therefore, tendrils should be easily visible in the potential vorticity contours, as is borne out by observations (Voropayev et al., 1999).

We notice from equation (3) that the two processes which govern the *potential vorticity* transport are advection and diffusion (we are ignoring the forcing for the moment). In the purely *diffusive process*, the potential vorticity diffuses throughout the domain in the direction of $-\nabla Q$, independent of the flow. In addition, it *advects*; potential vorticity is carried by fluid particles following the flow. Note that what we have discussed so far is an advective effect, which causes fluid to flow into (the case of a growing eddy) or out of (draining eddy) the eddy along a tendril. Perhaps paradoxically, our advective process is created through a *diffusive* splitting of the eddy boundary; diffusion and advection combine to create these tendrils. If one imagines (3) as expressed in nondimensional coordinates, it is easy to see that the pure diffusive effect is $\mathcal{O}(\epsilon)$ (occurs on a time-scale of size $1/\epsilon$), whereas the advection velocity is $\mathcal{O}(1)$ (not small). However, since the advective channel (the tendril) has size $\mathcal{O}(\epsilon)$, the advective *flux* of potential vorticity has size $\mathcal{O}(\epsilon)$ as well: it *also* occurs on a time-scale of $1/\epsilon$. Thus, the advective and diffusive fluxes of potential vorticity have the same magnitude. It is, therefore, unreasonable to ignore the advective effect in comparison with the purely diffusive effect when computing the potential vorticity balance for eddies.

A loose and intuitive understanding of tendrils could be that they result from the exterior portions of the eddy not being able to cope with the speed of rotation of the interior, with diffusivity providing retardation. However, our interpretation enables a more geometric explanation for tendrils. Diffusivity destroys the eddy boundary and creates a thin channel, along which an advective flux of potential vorticity occurs. This is not something which has been stated in the literature before; it is not an effect which can be ignored in comparison to pure diffusion of potential vorticity, with regard to eddy decay. This argument, in fact, works for any two-dimensional flow (not necessarily oceanographic) in which the relevant scalar quantity (not necessarily the potential vorticity) is subject to an advection-diffusion equation with small diffusivity. The physical cause of such diffusion may be the effects of small scale turbulence, viscosity, etc. Therefore, our Melnikov approach provides a pleasing possible explanation for eddy tendrils, as resulting from diffusivity breaking potential vorticity conservation.

Though the Melnikov approach provides information on how the manifolds perturb, it should be noted that the Melnikov function cannot describe how these manifolds behave after they wrap once around the homoclinic; the development is only valid for the first circuit of the manifolds around the homoclinic. Beyond this, the manifolds may wrap around and intersect in some complicated fashion, but any such effects would be at distances $\mathcal{O}(1)$ away from the unperturbed eddy boundary.

Should the eddy be defined by several saddle points on its boundary (rather than just one, as we have assumed in this paper), Melnikov functions would need to be calculated for each piece of the boundary which connects saddle points. However, the equations (9) and (10) cannot be applied in this situation, and need modification for the fact that heteroclinic trajectories (rather than homoclinic) form the separatrices of interest. A constant value for M_d is not obtained generically for this heteroclinic case, and the quick argument for tendril formation outlined above cannot be made. Therefore, it is not clear whether this explanation generalises to more complicated eddy boundaries.

7 Forcing contribution

We briefly consider how the forcing contribution of the Melnikov function (10) can be analysed via geometric conditions on the eddy boundary. Since $M_f(t)$ is dependent on t (unlike the diffusive contribution M_d), it is more difficult to obtain simple criteria for eddy growth. Therefore, we will only inspect the geometry under several restrictions, which are nevertheless of relevance in the Gulf Steam. Firstly, we shall specialise to standard models in which the flow is approximately steady in a eastward moving frame, as is commonly assumed (Pierrehumbert, 1991; del Castillo-Negrete and Morrison, 1993; Pratt et al., 1995). Then, we can set $c_2 = 0$ (i.e., $\eta = y$). Secondly, we shall assume that the additional forcing is steady and meridional, i.e. $f(x, y, t) = f(y)$ alone: a hypothesis which has been used in other oceanographic transport analyses in which the jet flow is mainly eastward (Poje and Haller, 1999). Thirdly, we shall suppose that the most southerly point of a warm eddy (or alternatively, the most northern point of a cold eddy) is its pinch-off point. This is a feasible assumption if addressing eddies in the process of pinching off from the Gulf Stream.

Under these conditions, the forcing contribution of the Melnikov function of (10) becomes

$$M_f = \int_{-\infty}^{\infty} \left[f\left(\bar{y}(\tau)\right) - f\left(0\right) \right] \, \mathrm{d}\tau,$$

where $(\bar{\xi}, \bar{y})$ is the parametrisation of the eddy boundary, η is identified precisely with y. We have replaced the moving frame forcing F with f, which additionally simplifies, since it has neither x nor t dependence. Under these simplifying assumptions, the forcing contribution is a constant and easy to analyse.

Consider the case, as usual, of a warm eddy. We have $\bar{y}(\tau) > 0$ for all τ, since the pinch-off point has y-coordinate 0, and is assumed to be the most southerly point on the eddy boundary. Then, if $f'(y) > 0$, $M_f > 0$ and the contribution shall be a growing one. On the other hand, if we take a cold eddy, we have $\bar{y}(\tau) < 0$ for all τ, and if $f'(y) > 0$, we would obtain $M_f < 0$: again the correct sign towards eddy growth. Thus, for both warm and cold eddies, *if $f'(y) > 0$ over the region of the eddy, then the eddy would be inclined to grow* under the influence of the forcing perturbation. Qualitatively speaking, *meridional forcing, which increases in the northward variable, contributes to eddy growth*. Even if $f'(y) < 0$ on some length of the eddy boundary, and if the length over which $f'(y) > 0$ is sufficiently large, we shall obtain the appropriate sign for eddy growth. As in our analysis of the previous section, we note that even if our qualitative condition(s) should not be satisfied exactly, the eddy may still grow if those qualities contribute sufficiently. We additionally stress that, in reality, it is the combined effect of *all* the contributions (diffusive and forcing) which give the eddy its growth instructions. Since the geometry changes dynamically, it is possible that the eddy grows at some times, and shrink at others.

8 Conclusions

This paper has analysed the qualitative structure of an eddy (Eulerian definition) which contributes towards its growth, and hence, its stability in a specific sense. All eddies of length scales greater than that corresponding to turbulent eddy diffusivities can be addressed through this viewpoint, which, therefore, includes both mesoscale and submesoscale eddies. The effective dynamics are considered a perturbation of potential vorticity conserving flow; the perturbation resulting directly from the inclusion of diffusion (and an additional small wind forcing) in the dynamical equation. Four qualitative observations were obtained in Sect. 5 which are indicative of eddy growth: (i) acute pinch-angle, (ii) small potential vorticity gradient across the eddy boundary, (iii) large radius of curvature of the eddy boundary, and (iv) the potential vorticity contours more tightly packed just within the eddy than outside. These conditions apply for both warm-core and cold-core eddies. If the wind forcing is meridional and steady, and the pinch-off point of the eddy is its most southernly (resp. most northernly) point for a warm (resp. cold) eddy, then another such contributory factor towards growth is that the wind forcing increases in the northward direction. The actual behaviour of the eddy depends upon the combination of all these factors.

Our eddy growth criteria are simple geometric conditions, which should be verifiable if potential vorticity data of a suitable resolution is available. The power of these conditions is that no knowledge of the velocity field is necessary. For the diffusive contributions, in fact, the criteria depend *only* on potential vorticity contours! The conditions are based, in reality, on the unperturbed potential vorticity contours, but for a 'nearly' potential vorticity conserving flow (such as believed to be true of oceanic jets and eddies); the perturbed contours could be expected to provide a sufficiently close approximation to the unperturbed ones. In any case, ours is a completely new approach to eddy stability in the presence of small diffusion, characterised by simple qualitative statements on the geometry of the potential vorticity field.

Paldor (1999), in analysing linear stability of discontinuous, radially symmetric, barotropic vortices, states in his abstract that if "the potential vorticity is continuous" at the boundary, "details of potential vorticity become important." Though in a different context, it is instructive that in our analysis of continuous potential vorticity models, it is exactly such details of potential vorticity contours which arise as conditions for eddy stability.

An added bonus from our arguments is that they give a possible explanation for tendrils which are often observed emanating from eddy structures. Diffusivity provides the mechanism for the breaking of the eddy boundary into a thin channel along the eddy boundary, where potential vorticity is *advected*. The potential vorticity contours develop a tendril along this channel as a result of the advection. Since the advection velocities are not small, these tendrils should be easily visible if diffusivity is present in the system. This is a generic effect for eddies with exactly one saddle point on their boundaries, and is to be expected whenever the conservation of a scalar field is broken through the inclusion of small diffusivity.

Acknowledgement. Conversations with Leonid Kuznetsov are gratefully acknowledged. Both authors were supported in part by the NSF through grant DMS-97-04906, and the ONR through grant N-00014-92-J-1481.

References

Balasuriya, S., Vanishing viscosity in the barotropic β-plane, J. Math. Anal. Appl., 214, 128–150, 1997.

Balasuriya, S., Jones, C. K. R. T., and Sandstede, B., Viscous perturbations of vorticity-conserving flows and separatrix splitting, Nonlinearity, 11, 47–77, 1998.

Biferale, L., Crisanti, A., Vergassola, M., and Vulpiani, A., Eddy diffusivities in scalar transport, Phys. Fluids, 7, 2725–2734, 1995.

Caflisch, R. and Sammartino, M., Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution, Commun. Math. Phys., 192, 463–491, 1998.

Brown, M. G. and Samelson, R. M., Particle motion in vorticity-conserving, two-dimensional incompressible flow, Phys. Fluids, 6, 2875–2876, 1994.

del Castillo-Negrete, D. and Morrison, P. J., Chaotic transport by Rossby waves in shear flow, Phys. Fluids A, 5, 948–965, 1993.

Dewar, W. K. and Gailliard, C., The dynamics of barotropically dominated rings, J. Phys. Oceanography, 24, 5–29, 1994.

Dewar, W. K. and Killworth, P. D., On the stability of oceanic rings, J. Phys. Oceanography, 25, 1467–1487, 1995.

Dewar, W. K., Killworth, P. D., and Blundell, J. R., Primitive-equation instability of wide oceanic rings. Part II: Numerical studies of ring stability, J. Phys. Oceanography, 29, 1744–1758, 1999.

Fannjiang, A. and Papanicolaou, G., Convection enhanced diffusion for periodic flows, SIAM J. Appl. Math., 54, 333–408, 1994.

Fenichel, N., Persistence and smoothness of invariant manifolds of flows, Indiana Univ. Math. J., 21, 193–226, 1971.

Flierl, G. R., On the instability of geostrophic vortices, J. Fluid Mech., 197, 349–388, 1988.

Guckenheimer, J. and Holmes, P., Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields, Springer, New York, 1983.

Haidvogel, D. B., Robinson, A. R., and Booth, C. G. H., Eddy induced dispersion and mixing, in Eddies in Marine Science, A. R. Robinson, (Ed.), Springer, Berlin, 1983.

Haller, G. and Poje, A. C., Eddy growth and mixing in mesoscale oceanographic flows, Nonlinear Processes in Geophys., 4, 223–235, 1997.

Helfrich, K. R. and Send, U., Finite-amplitude evolution of two-layer geostrophic vortices, J. Fluid Mech., 197, 331–348, 1988.

Hirsch, M. W., Pugh, C. C., and Shub, M., Invariant Manifolds (Lecture Notes in Mathematics 583), Springer, Berlin, 1977.

Jones, S. W., Interaction of chaotic advection and diffusion, Chaos Solitons Fractals, 4, 929–940, 1994.

Klapper, I., Shadowing and the role of small diffusivity in the chaotic advection of scalars, Phys. Fluids A, 4, 861–864, 1992.

McWilliams, J. C., Gent, P. R., and Norton, N. J., The evolution of balanced, low-mode vortices on the β-plane, J. Phys. Oceanography, 16, 838–855, 1986.

Mezić, I., Brady, J. F., and Wiggins, S., Maximal effective diffusivity for time-periodic incompressible flows, SIAM J. Appl. Math., 56, 40–56, 1996.

Miller, P., Jones, C. K. R. T., Rogerson, A. M., and Pratt, L. J., Quantifying transport in numerically generated velocity fields, Physica D, 110, 105–122, 1997.

Paldor, N., Linear instability of barotropic submesoscale coherent vortices observed in the ocean, J. Phys. Oceanography, 29, 1442–1452, 1999.

Pierrehumbert, R. T., Chaotic mixing of tracer and vorticity by modulated travelling waves, Geophys. Astrophys. Fluid Dyn., 58, 285–319, 1991.

Pedlosky, J., Geophysical Fluid Dynamics, Springer, New York, 1987.

Poje, A. C. and Haller, G., Geometry of cross-stream mixing in a double-gyre ocean model, J. Phys. Oceanography, 29, 1649–1665, 1999.

Poje, A. C., Haller, G., and Mezić, I., The geometry and statistics of mixing in aperiodic flows, Phys. Fluids, 11, 2963–2968, 1999.

Pratt, L. J., Lozier, M. S., and Beliakova, N., Parcel trajectories in quasi-geostrophic jets: neutral modes, J. Phys. Oceanography, 25, 1451–1466, 1995.

Richardson, P. L., Gulf Stream rings, in Eddies in Marine Science, Robinson, A.R. (ed.), Springer, Berlin, 1983.

Robinson, A. R., Overview and summary of eddy science, in Eddies in Marine Science, Robinson, A.R. (ed.), Springer, Berlin, 1983.

Rogerson, A. M., Miller, P. D., Pratt, L. J., and Jones, C. K. R. T., Lagrangian motion and fluid exchange in a barotropic meandering jet, J. Phys. Oceanography, 29, 2635–2655, 1999.

Rom-Kedar, V. and Poje, A. C., Universal properties of chaotic transport in the presence of diffusion, Phys. Fluids, 11, 2044–2057, 1999.

Voropayev, S. I., McEachern, G. B., Boyer, D. L., and Fernando, H. J. S., Experiment on the self-propagating quasi-monopolar vortex, J. Phys. Oceanography, 29, 2741–2751, 1999.

Weiss, J. B., Hamiltonian maps and transport in structured fluids, Physica D, 76, 230–238, 1994.

Weiss, J. B. and Knobloch, E., Mass transport and mixing by modulated travelling waves, Phys. Rev. A, 40, 2579–2589, 1989.